Mathematical Model of Forced Van Der Pol's Equation

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Abstract

This work is going to analyze the Forced Van Der Pol's Equation which is used to analyze the electric circuit. In 1927, Balthasar Van Der Pol observed the stable oscillation and heard some irregular noise in vacuum tube circuit. He then proposed the Forced Van Der Pol's Equation to analyze the circuit and suggested the concept of limit cycle and Chaos to explain his observation. In this work, We would like to analyze behavior of the model by observing the phase space, time series, bifurcation diagram and power spectrum. Those points in the figures are calculated by Runge-Kutta Method with the aid of MATLAB. For a better understanding, the RLC circuit, which is a electric circuit consisting the a resistor, an inductor and a capacitor, will be used as an example for explaining the properties. Apart from electric circuit, the Forced Van Der Pol's Equation can be applied to dynamic systems in different aspect, such as the artificial heart, economic market and so on. Therefore, we suggest this work to all students since the Forced Van Der Pol's Equation can be applied to many majors, like Mathematics, Physics, Economics, Sociology, Biology, Engineering and so on.

1 Introduction

Since the Forced Van Der Pol's Equation can be widely used in many aspects, we will analyze the model by observing the properties and behaviors in phase space, time series, bifurcation diagram and power spectrum, where those figures are generated by using MATLAB and the Runge-Kutta Method. We will demonstrate the stable oscillation (limit cycle) and irregular noise (Chaos) observed by Van Der Pol in those figures.

2 Literature Review

Since this work is reviewing and discussing the findings of the Forced Van Der Pol's Equation, this section will have a review of the figures and methods I used in this work.

2.1 The Runge-Kutta Method (RK4)

[4]The Runge-Kutta Method is a numerical analysis method which approximates the solution by repeating some calculation. The more iterative of calculation, the higher accuracy of solution will be generated. It is seldom used for manpower calculation since the iterative of calculation can be very complicated. [3]However, it is suitable for applying in this work because the iterative can be finished by MATLAB easily. Instead of solving the linear ODE, the problem we discuss about seems to be impossible to calculate manually. Therefore, RK4 is useful for solving non-linear ODE. This is a method that uses only linear transformation for the ODE

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function in approximation. Generally, the approximation is acceptable with a [4]small error. The Runge-Kutta Method will approximate $y(t)$ by the following recursion formula:

$$\begin{cases} k_1 = f(t_n, y_n) \\ k_2 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1) \\ k_3 = f(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2) \\ k_4 = f(t_n + h, y_n + hk_3) \\ t_{n+1} = t_n + h \\ y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{cases}$$

2.2 Phase space

[1]Phase space (also called phase diagram) is a useful diagram for dynamical systems, showing the relationship between domain x and the first derivative of domain $\dot{x}$. We can observe the property of fixed point and limit cycle. The stability can be divided into two types, stable and unstable. For the former, the fixed point will act as an attractor attracting all the trajectories. A region called basin of attraction may be recognized which indicates the region that the trajectories will go into certain fixed point. For the latter, the trajectories will come out from the fixed point and this case will not be discussed in this work. The method of determining the property of fixed point are as followed:
For a equivalent system,

$$\begin{cases} x^{'} = f(x,y) \\ y^{'} = g(x,y) \end{cases} \quad \text{, where } J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

we can calculate the Jacobian Matrix $J(x, y)$ and the value of $\tau = tr(J)$ and $\Delta = det(J)$. Then, the fixed point can be classified by comparing the value of τ, Δ and $\tau - 4\Delta$ with Table 1 below.

Table 1: Property of fixed point

	Saddle point	Stable node	Unstable node	Stable spiral	Unstable spiral
Δ	-	+	+	+	+
τ		-	+	-	+
$\tau^2 - 4\Delta$		+	+	-	-

2.3 Bifurcation Diagram

[2]Bifurcation diagram is a graph showing the relationship between a parameter and domain. In this work, we have plotted the domain x against the parameter ϵ. Knowing their relation, we can observe period of the model by drawing a vertical line on the figure. The number of intersection point of vertical line and curve in bifurcation diagram represent the number of period.In this work, we can observe the behavior of Chaos, which means there exist many period, in bifurcation diagram clearly.

2.4 Poincare Sections

[2]Poincare section is a figure similar as the bifurcation diagram, which shows the relationship between the domain x and its first derivative $\dot{x}$. Only one point in each period will be selected and plotted in the graph, so that we can observe the behavior of its period and property of Chaos.

2.5 Power Spectrum

Power Spectrum can demonstrate the property of existence of irregular noise, which indicate the number of period is directly proportional to the volume of noise. It shows the relationship between power spectral density $S(\omega)$ and frequency $\frac{\omega}{2\pi}$ of each term. The power spectral density are as followed

$$S(\omega) = \lim_{T\to\infty} \frac{\left|\hat{f}_T(\omega)\right|^2}{T} \,,$$

where $\hat{f}(\omega)$ is determined by *Fourier Transformation*. The property of Chaos and dominated frequency are also demonstrated in power spectrum.

3 Governing Equation

This section will discuss the governing equations. This is the Unforced Van Der Pol's Equation,

$$\ddot{x} - \frac{\epsilon}{\omega}(1 - x^2)\dot{x} + x = 0 \,, \tag{1}$$

where x is representing the voltage at one point in the RLC circuit varying with time t. Although the value of x is not exactly equal to the value of voltage, they are directly proportional to each other, we can still observe the behavior of voltage from x. Thus, $\dot{x} = \frac{\partial x}{\partial t}$ and $\ddot{x} = \frac{\partial^2 x}{\partial t^2}$ are studying the variation of voltage in different time. There are two parameters ϵ and ω which are assumed to be positive since we are only concerned with the case of positive parameters. The term $\frac{\epsilon}{\omega}(1 - x^2)\dot{x}$ represents the damping, which means the resistance decreasing the voltage. Adding the external driven voltage (forced) $\epsilon F cos\omega t$, which is a periodic increasing and decreasing the value of x with different frequency, we have the Forced Van Der Pol's Equation

$$\ddot{x} - \frac{\epsilon}{\omega}(1 - x^2)\dot{x} + x = \epsilon F cos\omega t \,, \tag{2}$$

where F is another positive parameter and the unforced case is modeled by letting $F = 0$. Then, let $y = \dot{x}$ and we can rewrite the equation (2) into equivalent system and get the full model of Van Der Pol's Equation

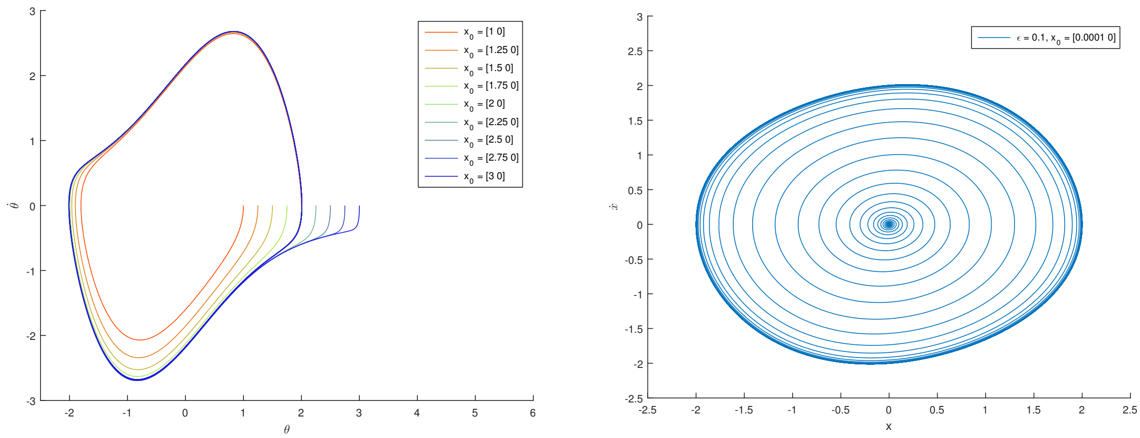
$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} y \\ \epsilon F\cos\omega t + \frac{\epsilon}{\omega}(1 - x^2)y - x \end{pmatrix}, \qquad \text{where} \quad y = \dot{x}.$$

4 Results

This section will present our findings using phase space, bifurcation diagram, poincare section and power spectrum. Before discussing the forced case, for simplicity we set $F = 0$ and consider the unforced case first.

4.1 Unforced Van Der Pol's Equation

From equation (2) , We can observe the property of oscillation from the damping force term $\frac{\epsilon}{\omega}(1-x^2)\dot{x}$. When $| x |> 1$, the function $(1-x^2)$ will be negative which implies that the damping force is positive. Thus, the positive damping force will decrease the value of x. When $| x |< 1$, the damping force will be negative which increases the value of x. In short, the value of x will be increased when it is smaller than one and decreased when it is larger than one. The behavior will repeat again and again which suggests there is the oscillation property of x. It suggests that in the case of electric circuit, voltage will also demonstrate the oscillation behavior.
In Figure 1(a), the Phase Space shows the trajectories going into the closed cycle when time passes for different initial condition. Both the trajectories inside and outside the limit cycle will



(a) When $\omega = 1$ and $\epsilon = 1$ (b) When $\omega = 1$ and $\epsilon = 0.1$

Figure 1: Phase Space with different initial conditions

go into the limit cycle. It suggests that whatever the initial value of voltage is, the voltage will demonstrate the same periodic behavior after some time for the same parameters. It is trapped in the closed cycle called the limit cycle. During the limit cycle, the energy conserved, which implies that the total energy in the circuit will be unchanged. In Figure 1(b), the limit cycle even shows a beautiful circular limit cycle for small initial condition. The trajectories come out from the fixed point to the limit cycle with a spiral pattern.

Remarks: There are two types of limit cycle, one is stable and one is unstable. The trajectories will go to the limit cycle when time approaches positive infinity for the stable case. On the contrary, the limit cycle will go to the limit cycle when time approaches negative infinity for the unstable case.

We can observe there exist a spiral pattern inside limit cycle in figure 1(b). However, this is not the only case. There are different properties of fixed points for different parameters. Finding the property of fixed point, we will further simplify the equation by letting $\mu = \frac{\epsilon}{\omega}$

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0\,, \tag{3}$$

and the equivalent system will be written as

$$\begin{aligned} y &= \dot{x} \\ \dot{y} &= \mu(1 - x^2)\dot{x} - x\,. \end{aligned} \tag{4}$$

We now have the Jacobian Matrix, thus, the eigenvalue

$$J = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \qquad \Rightarrow \lambda_{\pm} = \frac{1}{2}(\mu \pm \sqrt[2]{\mu^2 - 4})\,.$$

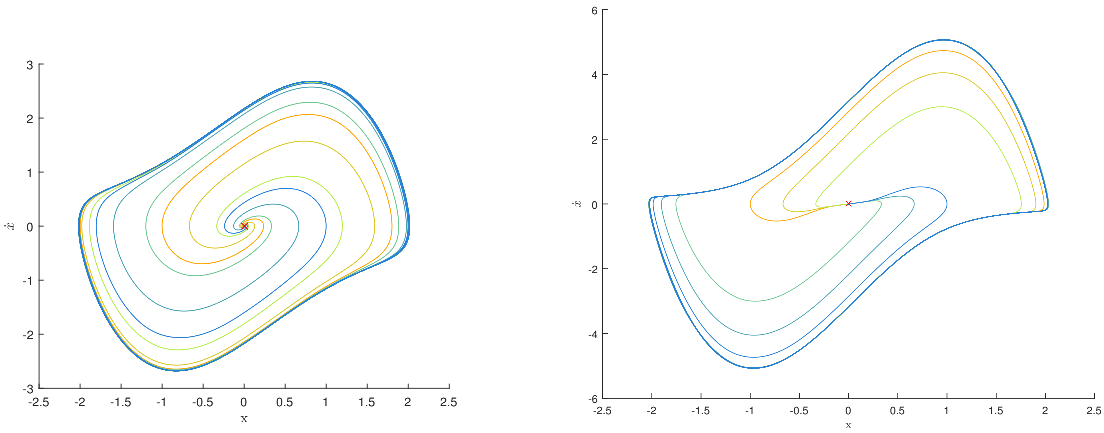
According to Table 1, the property of fixed point can be determined. Table 2 shows the properties of fixed point for different μ, we can observe that when μ is negative, the fixed point will be unstable implying that the trajectories inside the limit cycle will leave the limit cycle and go into the fixed point. Similarly, trajectories outside the limit cycle will leave the limit cycle. That is the reason why the negative parameters are not analyzed. When μ is positive, the fixed point will have stable properties which the trajectories will escape the fixed point and will be trapped in the limit cycle after some time. It will demonstrate the unstable spiral and unstable node properties when $\mu \in (0, 2)$ and $\mu \in (2, \infty)$ respectively, which are shown in Figure 2 .

When $\mu = 0$, the equation will be changed into a Simple Harmonic Oscillator,

$$\ddot{x} + x = 0 \tag{5}$$

Table 2: Property of fixed point for different μ

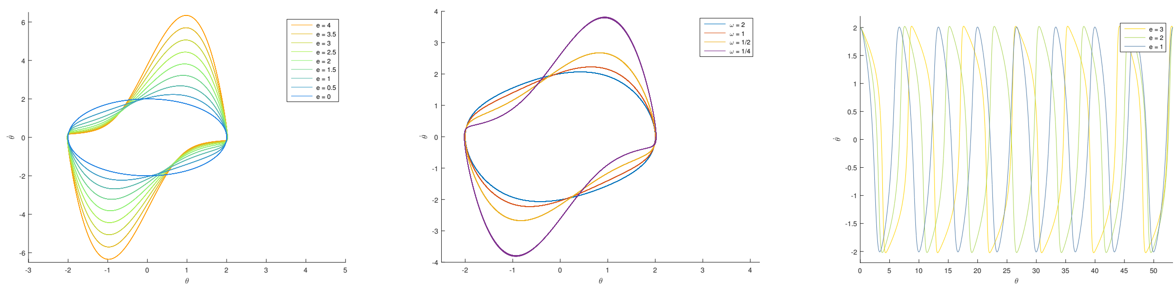
μ	τ	Δ	$\tau - 4\Delta$	Property
(0,2)	+	+	-	Unstable spiral
(-2,0)	-	+	-	Stable spiral
(2,∞)	+	+	+	Unstable node
($-\infty$,-2)	-	+	+	Stable node
-2	-	+	0	Undefined
0	0	+	-	Elliptic
2	+	+	0	Undefined



(a) When $\mu = 1 \in (0, 2)$ (b) When $\mu = 3 \in (2, \infty)$

Figure 2: Phase Space for different value

where the solution will be elliptical and this property is shown in the Figure 3(a). We can observe when ϵ is getting smaller, the shape of phase space will be more like a elliptic which means the property of this model will act more like a Simple Harmonic Oscillator. In Figure 3(c) , the shape is more like a elliptic as ω gets larger. We can also conclude that the period will increase when ϵ gets smaller and ω gets larger. This kind of property can be shown in Figure 3(c) , showing that the period is larger when ϵ increases.

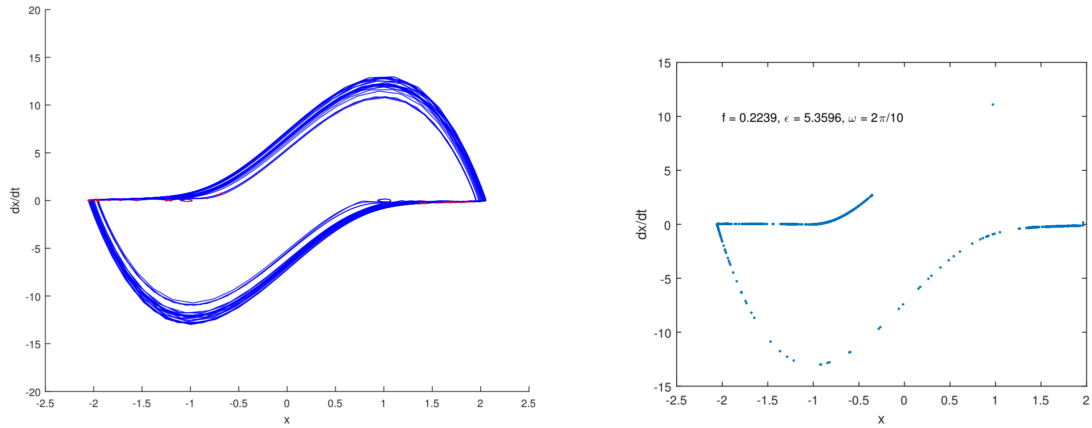


(a) Phase Diagram of different value of ϵ (b) Phase Diagram of different value of ω (c) Time Series of different value of ϵ

Figure 3: Examples of Fractals

4.2 Forced Van Der Pol's Equation

For the forced case, we consider the full model of Van Der Pol's Equation by equation (2). Periodic motion still can be found under periodic force, although the frequency of the external driven force $\epsilon F cos\omega t$ is not equal to the circuit oscillation period in some cases. However, we are interested in Chaos under external driven force.
In the Poincare Section, we can clearly discover that the chaotic motion might occur under certain parameters. One of the Chaos is shown in Figure 4. Note that the phase spaces when chaotic motion occurs still remain a closed curve. It is mainly related to the property of Van Der Pol's Equation: limit cycle and the tendency of energy stable. Therefore, the phase space and the Poincare Section still demonstrate a "double-S" shape.

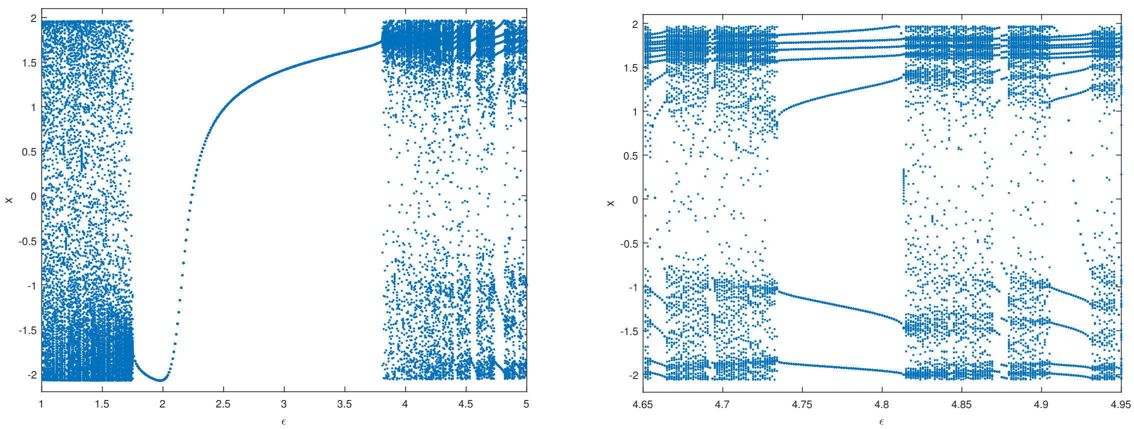


(a) Phase Space (Red points represent the corresponding coordinate of Poincare Section)

(b) Poincare Section

Figure 4: Different Graph under Chaotic motion

In the bifurcation diagram, we can clearly see that chaotic and periodic motions can also happen. 1-period, 7-period, and Chaotic motion can be found when $\epsilon = 2, 4.75, 5.54$, respectively. The parameters of the Figure 4(b) are also picked from Figure 5(b). Another observation is that the points of bifurcation are rarely distributed in $x \in [-0.5, 0.5]$. It can be related to the properties of this equation: limited cycle and the tendency of energy stable.



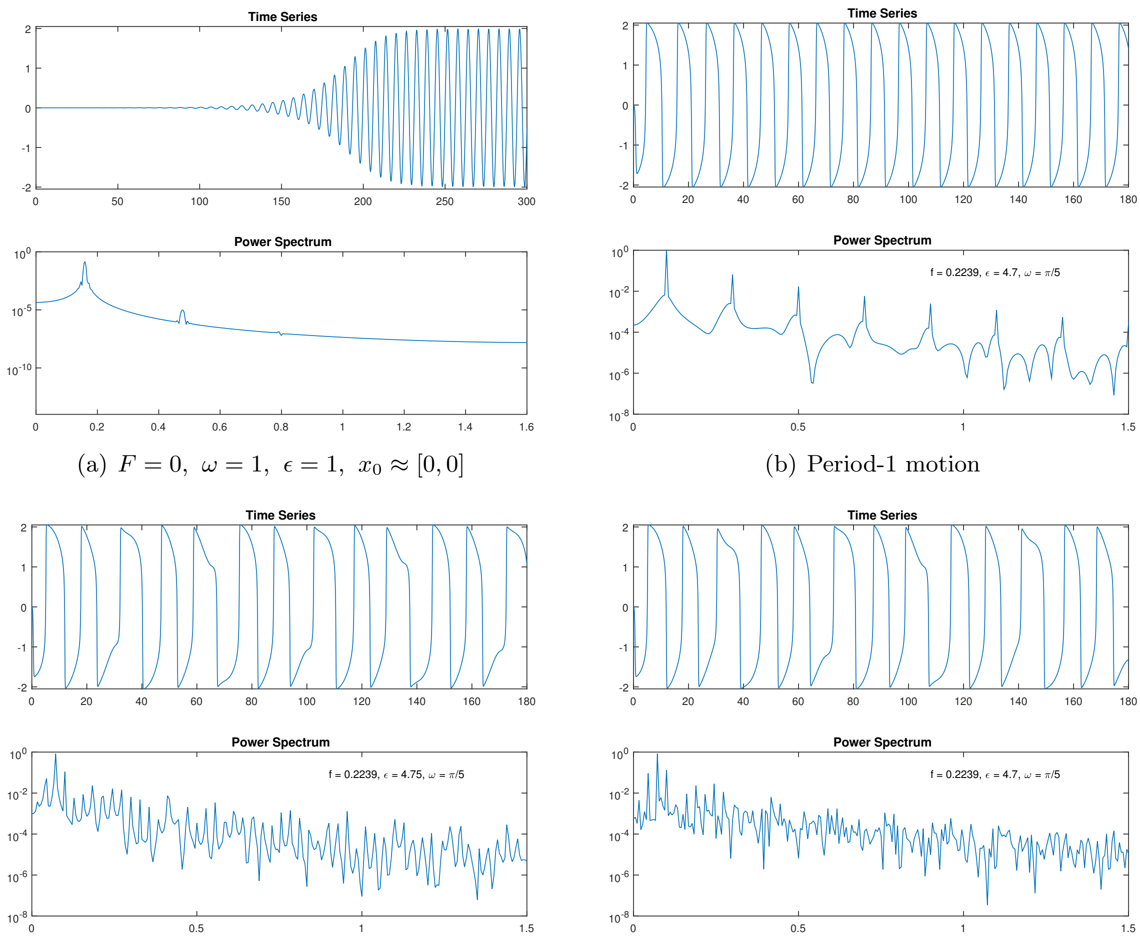
(a) $\epsilon \in [1, 5]$

(b) $\epsilon \in [4.65, 4.95]$

Figure 5: Bifurcation Diagram when $F = 0.2239, \quad \omega = 2\pi/10$

In the Power Spectrum, we observed that the spectrum is a smooth curve when the Van Der

Pol's Oscillator is under undriven, which is consist when the result of the oscillate is periodic, which was shown in Figure 6(a). It is hard to determine the number of period motion in Power Spectrum when it turns larger. However we were still able to notice more noise will be made when motion is Chaotic, shown in Figure 6(d) .



(a) $F = 0,\ \omega = 1,\ \epsilon = 1,\ x_0 \approx [0, 0]$

(b) Period-1 motion

(c) Period-7 motion

(d) Chaotic motion

Figure 6: Time Series and the corresponding Power Spectrum under different parameters

5 Conclusion and Discussion

Van Der Pol's Equation was applied on different aspect of reality, which is not limited in Physics or Mathematics. Van Der Pol's Equation can be applied to some economics model to explain the Chaos of the economics market in theory. The medical device which requires high periodic stability like artificial heart relies on this equation to predict the Chaos of the output voltage for adjusting the appropriate voltage. Therefore, we suggest this work to all students since the Forced Van Der Pol's Equation can be applied to many majors, like Mathematics, Physics, Economics, Sociology, Biology, Engineering and so on.

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