ON THE QUANTUM DIFFERENTIATION OF SMOOTH REAL-VALUED FUNCTIONS

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Abstract. Calculating the value of $C^{k\in\{1,\infty\}}$ class of smoothness real-valued function's derivative in point of $\mathbb{R}^+$ in radius of convergence of its Taylor polynomial (or series), applying an analog of Newton's binomial theorem and q-difference operator. (P,q)-power difference introduced in section 5. Additionally, by means of Newton's interpolation formula, the discrete analog of Taylor series, interpolation using q-difference and p,q-power difference is shown.

Keywords. Derivative, Differential calculus, Differentiation, Taylor's theorem, Taylor's formula, Taylor's series, Taylor's polynomial, Power function, Binomial theorem, Smooth function, Newton's interpolation formula, Finite difference, Q-derivative, Jackson derivative, Q-calculus, Quantum calculus, Q-difference, Quantum algebra

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1. Introduction

Let be Taylor's theorem (see §7 "Taylor's formula", [1])

Theorem 1.1. ***Taylor's theorem.*** *Let be $n \geq 1$ an integer, let function $f(x)$ be $n+1$ times differentiable in neighborhood of $a \in \mathbb{R}$. Let x be an any function's argument from such neighborhood, p - some positive number. Then, there is exist some c between points a and x, such that*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x) \tag{1.2}$$

where $R_{n+1}(x)$ - general form of remainder term

$$R_{n+1}(x) = \left(\frac{x-a}{x-a}\right)^p \frac{(x-c)^{n+1}}{n!p} f^{(n+1)}(c) \tag{1.3}$$

Proof. Denote $\varphi(x,a)$ polynomial related to x of order n, from right part of (1.2), i.e

$$\varphi(x,a) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{1.4}$$

Next, denote as $R_{n+1}(x)$ the difference

$$R_{n+1}(x) = f(x) - \varphi(x,a) \tag{1.5}$$

Theorem will be proven, if we will find that $R_{n+1}(x)$ is defined by (1.3). Let fix some x in neighborhood, mentioned in theorem 1.1. By definition, let be $x > a$. Denote by t an variable, such that $t \in [a,x]$, and review auxiliary function $\psi(t)$ of the form

$$\psi(t) = f(x) - \varphi(x,t) - (x-t)^p Q(x) \tag{1.6}$$

where

$$Q(x) = \frac{R_{n+1}(x)}{(x-a)^p} \tag{1.7}$$

More detailed $\psi(t)$ could be written as

$$\psi(t) = f(x) - f(t) - \frac{f'(t)}{1!}(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - (x-t)^p Q(x) \tag{1.8}$$

Our aim - to express $Q(x)$, going from properties of introduced function $\psi(t)$. Let show that function $\psi(t)$ satisfies to all conditions of Rolle's theorem [2] on $[a,x]$. From (1.8) and conditions given to function $f(x)$, it's obvious, that function $\psi(t)$ continuous on $[a,x]$. Given $t = a$ in (1.6) and keeping attention to equality (1.7), we have

$$\psi(a) = f(x) - \varphi(x,a) - R_{n+1}(x) \tag{1.9}$$

Hence, by means of (1.5) obtain $\psi(a) = 0$. Equivalent $\psi(x) = 0$ immediately follows from (1.8). So, $\psi(t)$ on segment $[a,x]$ satisfies to all necessary conditions of Rolle's theorem [2]. By Rolle's theorem, there is exist some $c \in [a,x]$, such that

$$\psi'(c) = 0 \tag{1.10}$$

Calculating derivative $\psi'(t)$, differentiating equality (1.8), we have

$$\psi'(t) = -f'(t) + \frac{f'(t)}{1!} - \frac{f''(t)}{2!}(x-t) + \frac{f''(t)}{2!}2(x-t) - \cdots + \frac{f^{(n)}(t)}{n!}n(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n + p(x-t)^{p-1}Q(x) \tag{1.11}$$

It's seen that all terms in right part of (1.11), except last two items, self-destructs. Hereby,

$$\psi'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + p \cdot (x-t)^{p-1}Q(x) \tag{1.12}$$

Given $t = c$ in (1.12) and applying (1.10), obtain

$$Q(x) = \frac{(x-c)^{n-p+1}}{n!p} f^{(n+1)}(c) \tag{1.13}$$

By means of (1.13) and (1.7), finally, we have

$$R_{n+1}(x) = (x-a)^p Q(x) = \left(\frac{x-a}{x-a}\right)^p \frac{(x-c)^{n+1}}{n!p} f^{(n+1)}(c) \tag{1.14}$$

Case $x < a$ is reviewed absolutely similarly. (see for reference [1], pp 246-247) This proves the theorem. □

Let function $f(x) \in C^k$ class of smoothness and satisfies to theorem 1.1, then its derivative by means of its Taylor's polynomial centered at $a \in \mathbb{R}$ in radius of convergence with $f(x)$ and linear nature of derivative, $(gf(x)+um(x))^{'} = gf^{'}(x)+um^{'}(x)$, is

$$\frac{d}{dx}f(x) = \frac{f'(a)}{1!}\frac{d}{dx}(x-a) + \frac{f''(a)}{2!}\frac{d}{dx}(x-a)^2 + \cdots + \frac{f^{(k-1)}(a)}{(k-1)!}\frac{d}{dx}(x-a)^k + R'_{k+1}(x) \tag{1.15}$$

Otherwise, if $f \in C^\infty$ we have derivative of Taylor series [5] of f given the same conditions as (1.15)

$$\frac{d}{dx}f(x) = \frac{f'(a)}{1!}\frac{d}{dx}(x-a) + \frac{f''(a)}{2!}\frac{d}{dx}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}\frac{d}{dx}(x-a)^n + + \cdots \tag{1.16}$$

Hence, derivative of function $f : 1 \leq C(f) \leq \infty$ [1] could be reached by differentiating of its Taylor's polynomial or series in radius of convergence, and consequently summation of power's derivatives being multiplied by coefficient, according theorem 1.1, over k from 1 to $t \leq \infty$, depending on class of smoothness. Hereby, the properties of power function's differentiation holds, in particular, the derivative of power close related to Newton's binomial theorem [4].

Lemma 1.17. *Derivative of power function equals to limit of Binomial expansion of* $(x+\Delta x)^n$*, iterated from* 1 *to* n*, divided by* $\Delta x : \Delta x \to 0$.

Proof.

$$\frac{d(x^n)}{dx} = \lim_{\Delta x \to 0} \left\{ \sum_{k=1}^{n} \binom{n}{k} x^{n-k} (\Delta x)^{k-1} \right\} = \binom{n}{1} x^{n-1} \tag{1.18}$$

□

According to lemma (1.17), Binomial expansion is used to reach derivative of power, otherwise, let be introduced expansion, based on forward finite differences, discussed in [3]

$$x^n = x^{n-2} + j \sum_{k \in \mathfrak{C}(x)} k \cdot x^{n-2} - k^2 \cdot x^{n-3}, \quad x \in \mathbb{N} \tag{1.19}$$

[1]For example, let f be a k-smooth function, then $C(f) = k$.

where $j = 3!$ and $\mathfrak{C}(x) := \{0,\ 1,\ \ldots,\ x\} \subseteq \mathbb{N}$. Particularize[2] (1.19), one has

$$x^n = \sum_{k \in \mathfrak{U}(x)} j \cdot k \cdot x^{n-2} - j \cdot k^2 \cdot x^{n-3} + x^{n-3} \tag{1.20}$$

where $\mathfrak{U}(x) := \{0,\ 1,\ \ldots,\ x-1\} \subseteq \mathbb{N}$.

Property 1.21. *Let $\mathfrak{S}(x)$ be a set $\mathfrak{S}(x) := \{1,\ 2,\ \ldots,\ x\} \subseteq \mathbb{N}$, let be (1.20) written as $T(x,\ \mathfrak{U}(x))$, then we have equality*

$$T(x,\ \mathfrak{U}(x)) \equiv T(x,\ \mathfrak{S}(x)),\ x \in \mathbb{N} \tag{1.22}$$

Let (1.19) be denoted as $U(x,\ \mathfrak{C}(x))$, then

$$U(x,\ \mathfrak{C}(x)) \equiv U(x,\ \mathfrak{S}(x)) \equiv U(x,\ \mathfrak{U}(x)) \tag{1.23}$$

Proof. Let be a plot of $jkx^{n-2} - jk^2x^{n-3} + x^{n-3}$ by k over $\mathbb{R}^+_{\leq 10}$, given $x = 10$

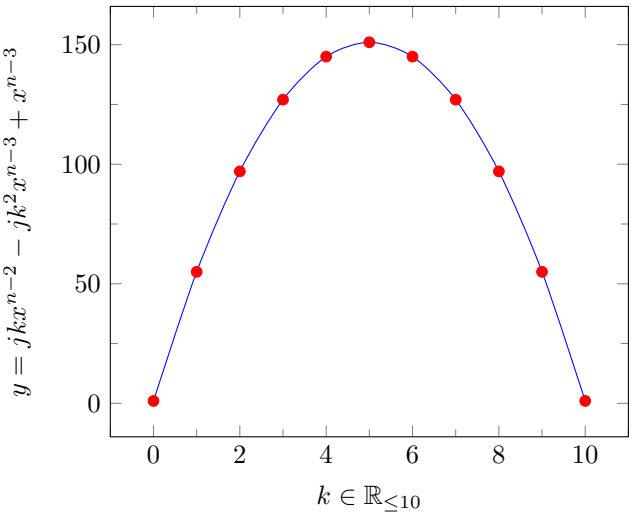

Figure 1. Plot of $jkx^{n-2} - jk^2x^{n-3} + x^{n-3}$ by k over $\mathbb{R}^+_{\leq 10}$, $x = 10$

Obviously, being a parabolic function, it's symmetrical over $\frac{x}{2}$, hence equivalent $T(x,\ \mathfrak{U}(x)) \equiv T(x,\ \mathfrak{S}(x)),\ x \in \mathbb{N}$ follows. Reviewing (1.19) and denote $u(t) = tx^{n-2} - t^2x^{n-3}$, we can make conclusion, that $u(0) \equiv u(x)$, then equality of $U(x,\ \mathfrak{C}(x)) \equiv U(x,\ \mathfrak{S}(x)) \equiv U(x,\ \mathfrak{U}(x))$ immediately follows.
This completes the proof. □

By definition we will use set $\mathfrak{U}(x) \subseteq \mathbb{N}$ in our next expressions.
Since, for each $x = x_0 \in \mathbb{N}$ we have equivalent

Lemma 1.24. *$\forall x = x_0 \in \mathbb{N}$ holds*

$$\underbrace{\sum_{t=1}^{x} \sum_{k=1}^{n} \binom{n}{k} t^{n-k}}_{x^n} \equiv \sum_{k=0}^{x-1} j \cdot k \cdot x^{n-2} - j \cdot k^2 \cdot x^{n-3} + x^{n-3} \tag{1.25}$$

[2]Transferring x^{n-2} under sigma operator, decreasing the power by 1 and taking summation over $k \in \mathfrak{U}(x)$

Proof. Proof can be done by direct calculations. □

By lemma 1.24 we have right to substitute (1.20) into limit (1.18), replacing Binomial expansion, and represent derivative of power by means of expression (1.20). Note that,

$$\Delta(x^n) = \sum_{k=1}^{n} \binom{n}{k} x^{n-k} \neq j \cdot k \cdot x^{n-2} - j \cdot k^2 \cdot x^{n-3} + x^{n-3} \tag{1.26}$$

As (1.20) is analog of Binomial expansion of power and works only in space of natural numbers, different in sense, that Binomial expansion, for example, could be denoted as $M(x,\ \mathfrak{C}(n))$, where n - exponent. While (1.20) could be denoted $T(x,\ \mathfrak{U}(x) \equiv \mathfrak{S}(x))$, it shows that in case of Binomial expansion the set over which we take summation depends on exponent n of initial function, when for (1.20) it depends on point $x = x_0 \in \mathbb{N}$. To provide expressions' (1.20) usefulness[3] on taking power's derivative over $\mathbb{R}^+$, derivative in terms of quantum calculus should be applied, as next section dedicated to.

2. Application of Q-Derivative

Derivative of the function f defined as limit of division of function's grow rate by argument's grow rate, when grow rate tends to zero, and graphically could be interpreted as follows

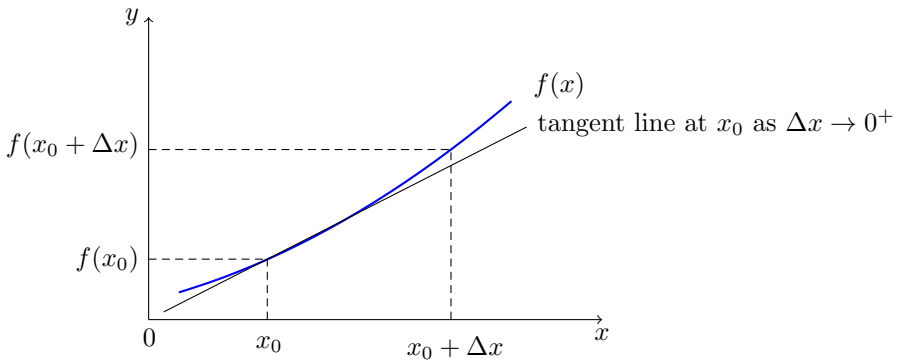

Figure 2. Geometrical sense of derivative

In 1908 Jackson [10] reintroduced [11], [12] the Euler-Jackson q-difference operator [9]

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad x \neq 0 \tag{2.1}$$

The limit as q approaches 1^- is the derivative

$$\frac{df}{dx} = \lim_{q \to 1^-} (D_q f)(x) \tag{2.2}$$

[3]By classical definition of derivative, we have to use upper summation bound $(x + \Delta x) \in \mathbb{R}^+$ on (1.20), which turns false result as (1.20) works in space of $\mathbb{N}$.

More generalized form of q-derivative

$$\frac{df(x)}{dx} = \lim_{q\to 1^-} \underbrace{\frac{f(x)-f(xq)}{x-xq}}_{(D_q f^-)(x)} \equiv \lim_{q\to 1^+} \underbrace{\frac{f(xq)-f(x)}{xq-x}}_{(D_q f^+)(x)} \tag{2.3}$$

where $(D_q f^+)(x)$ and $(D_q f^-)(x)$ forward and backward q-differences, respectively. The follow figure shows the geometrical sense of above equation as q tends to 1^+

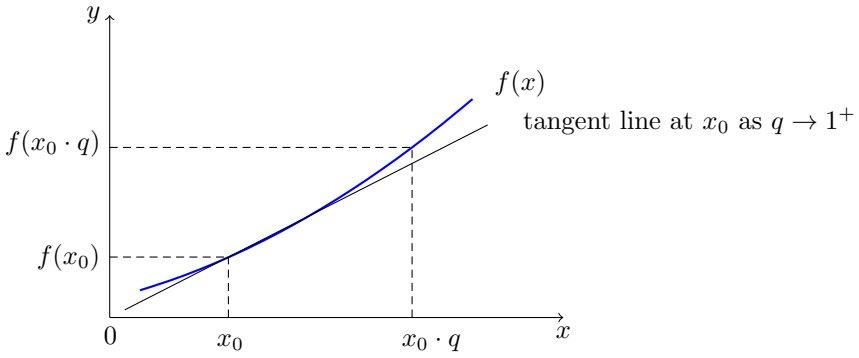

Figure 3. Geometrical sense of right part of (2.3)

Review the monomial x^n, where n-positive integer and applying right part of (2.3), then in terms of q-calculus we have forward q-derivative over $\mathbb{R}$

$$\frac{d(x^n)}{dx} = \lim_{q\to 1^+} (D_q x^{n^+})(x) = \lim_{q\to 1^+} \frac{x^n(q^n-1)}{x(q-1)} \tag{2.4}$$

$$= \lim_{q\to 1^+} x^{n-1} \sum_{k=0}^{n-1} q^k, \quad q \in \mathbb{R}$$

Otherwise, see reference [9], equation (109).
Generalized view of high-order power's derivative by means of (2.4)

$$\frac{d^k(x^n)}{dx^k} = \lim_{q\to 1^+} (D_q^k x^{n^+})(x) = \lim_{q\to 1^+} x^{n-k} \prod_{j=0}^{k-1} \left(\sum_{m=0}^{n-j} q^m \right) \tag{2.5}$$

Since, the main property of power is

Property 2.6.

$$(x \cdot y)^n = x^n \cdot y^n$$

Let be definition

Definition 2.7. By property (2.6) and (1.20), definition of $c = x \cdot t : t \in \mathbb{R},\ x \in \mathbb{N} \Rightarrow c \in \mathbb{R}$ to power $n \in \mathbb{N}$

$$c^n := \xi(x,\ t)_n := \sum_{k=0}^{x-1} jkx^{n-2} \cdot t^n - jk^2 x^{n-3} \cdot t^n + x^{n-3} \cdot t^n \tag{2.8}$$

Hereby, applying definition (2.7) and (2.4), derivative of monomial $x^n : n \in \mathbb{N}$ by x in point $x_0 \in \mathbb{N}$ is

$$\frac{d(x^n)}{dx}\bigg|_{x=x_0} = \underbrace{\lim_{q\to 1^+}\frac{\xi(x,\ q)_n - \xi(x,\ 1)_n}{x\cdot q - x}}_{\stackrel{\text{def}}{=}\mathscr{D}_{q>1}[x^n]} \equiv \underbrace{\lim_{q\to 1^-}\frac{\xi(x,\ 1)_n - \xi(x,\ q)_n}{x\cdot q - x}}_{\stackrel{\text{def}}{=}\mathscr{D}_{q<1}[x^n]}, \tag{2.9}$$

Let us approach to extend the definition space of expression (2.9) from $x_0 \in \mathbb{N}$ to $x_0 \in \mathbb{R}^+$. Let be $x_0 = \xi(t_0,\ p)_1 \in \mathbb{R}^+ \not\supseteq \mathbb{N}$ as $p \in \mathbb{R}^+ \not\supseteq \mathbb{N}$ and $t_0 \in \mathbb{N}$, then applying (p,q)-difference discussed in [13]

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x},\ x \neq 0 \tag{2.10}$$

by means of definition (2.7) and (2.10), (p,q)-differentiating of monomial $x^n,\ n \in \mathbb{N}$ gives us

$$\frac{d(x^n)}{dx}\bigg|_{x=t_0} = \lim_{p\to q^+} D_{p,q}x^n = \underbrace{\lim_{p\to q^+}\frac{\xi(x,\ p)_n - \xi(x,\ q)_n}{x\cdot p - x\cdot q}}_{\stackrel{\text{def}}{=}\mathscr{D}_{p\to q}[x^n]} \tag{2.11}$$

$$\equiv \underbrace{\lim_{q\to p^-}\frac{\xi(x,\ p)_n - \xi(x,\ q)_n}{x\cdot p - x\cdot q}}_{\stackrel{\text{def}}{=}\mathscr{D}_{p\leftarrow q}[x^n]},\quad t_0 \in \mathbb{N},\ [p,\ q] \in \mathbb{R}^+ \not\supseteq \mathbb{N}$$

Geometrical interpretation is shown below

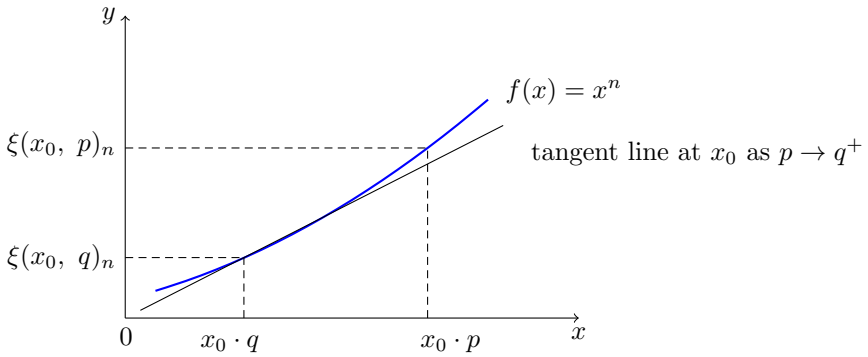

Figure 4. Geometrical interpretation of (2.11)

3. Application on functions of finite class of smoothness

In this section we will get derivative of function $f \in C^n$ in point $x_0 \in \mathbb{R}^+$ by means of its Taylor's polynomial and (2.9), where n - some positive integer. Let $f(x)$ be an n-smooth function, then derivative of its Taylor's polynomial at radius of convergence with f in $x_0 : (x_0 - a) \in \mathbb{N}$ is

$$\frac{df(x)}{dx}\bigg|_{x=x_0} = \sum_{k=1}^{n}\left[\frac{f^{(k)}(a)}{k!}\mathscr{D}_{q>1}[(x-a)^k]\right] + \mathscr{D}_{q>1}[R_{n+1}(x)] \tag{3.1}$$

$$\equiv \sum_{k=1}^{n}\left[\frac{f^{(k)}(a)}{k!}\mathscr{D}_{q<1}[(x-a)^k]\right] + \mathscr{D}_{q<1}[R_{n+1}(x)]$$

Otherwise, let $(x_0 - a)$ satisfies to conditions of (2.11), i.e $(x_0 - a) \in \mathbb{R}^+$, then applying operator $\mathscr{D}$, defined in (2.9) we can reach derivative of $f : f \in C^n$ in point $x_0 : (x_0 - a) \in \mathbb{R}^+$, by differentiation of its Taylor's polynomial in radius of convergence with f, that is

$$\frac{df(x)}{dx}\bigg|_{x=t_0} = \sum_{k=1}^{n}\left[\frac{f^{(k)}(a)}{k!}\mathscr{D}_{p\to q}[(x-a)^k]\right] + \mathscr{D}_{p\to q}[R_{n+1}(x)] \tag{3.2}$$

$$\equiv \sum_{k=1}^{n}\left[\frac{f^{(k)}(a)}{k!}\mathscr{D}_{p\leftarrow q}[(x-a)^k]\right] + \mathscr{D}_{p\leftarrow q}[R_{n+1}(x)]$$

4. Application on analytic functions

If $f \in C^\infty$ (i.e analytic), then approximation by means of Taylor series holds in neighborhood of its center at $a \in \mathbb{R}$. Suppose that f is real-valued and satisfies to conditions of Taylor's theorem 1.1, then derivative of f at $x_0 : x < x_0 < a$ is

$$\frac{df(x)}{dx} = \frac{d}{dx}\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k = \left[\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}\frac{d}{dx}(x-a)^k\right]_{x=x_0} \tag{4.1}$$

Let x_0 satisfies to conditions of (3.1), then, applying definition (2.7), we have derivative of f in point $x_0 \in \mathbb{R}^+$

$$\frac{df(x)}{dx} = \left[\sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}\mathscr{D}_{q>1}[(x-a)^k] \equiv \sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}\mathscr{D}_{q<1}[(x-a)^k]\right]_{x=x_0} \tag{4.2}$$

Otherwise, if x_0 satisfies to conditions of (3.2) and x_0 in radius of convergence with f, then derivative of $f \in C^\infty$, by means of its Taylor's series and (2.7), is

$$\frac{df(x)}{dx} = \left[\sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}\mathscr{D}_{p\to q}[(x-a)^k] \equiv \sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}\mathscr{D}_{p\leftarrow q}[(x-a)^k]\right]_{x=t_0} \tag{4.3}$$

5. Introduction of $(P,\ q)$-power difference

Lemma 5.1. *Let be $m \in \mathbb{R}/\mathbb{I}$ and m could be represented as $m = at$, then exists some $c \in \mathbb{R}/\mathbb{I}$, such that*

$$m = a^c \tag{5.2}$$

Reviewing (2.3), we can see, that argument's differential Δx is given by $x \cdot q - x$, according to lemma 5.1 $\exists c \in \mathbb{R}/\mathbb{I},\ x \cdot t - x = x^c - x$, then, from (2.4) immediately follows q-power difference, (see [14], page 2, equation 3)

$$\mathcal{D}_{q>1}f(x) := \frac{f(x^q) - f(x^1)}{x^q - x^1}, \quad x \neq 0 \tag{5.3}$$

As q tends to 1^+ we have reached derivative

$$\frac{df(x)}{dx} = \lim_{q\to 1^+}\mathcal{D}_{q>1}f(x) = \underbrace{\lim_{q\to 1^+}\frac{f(x^q) - f(x^1)}{x^q - x^1}}_{\stackrel{\text{def}}{=}\mathbf{D}_{q>1}[f(x)]} \tag{5.4}$$

$$\equiv \underbrace{\lim_{q\to 1^-}\frac{f(x^1)-f(x^q)}{x^1-x^q}}_{\stackrel{\text{def}}{=}\mathbf{D}_{q<1}[f(x)]} =: \lim_{q\to 1^-}\mathcal{D}_{q<1}f(x)$$

where $\lim\limits_{q\to 1^-}\mathcal{D}_{q<1}f(x)$ denotes the derivative through backward q-power difference By lemma 5.1 from (2.10) immediately follows (p,q)-power difference

$$\mathcal{D}_{p\to q}f(x) := \frac{f(x^p)-f(x^q)}{x^p-x^q}, \quad x\neq 0 \tag{5.5}$$

Hence, for $v=x^p,\ p\in\mathbb{R}$

$$\frac{df(x)}{dx}(v) = \lim_{p\to q^+}\mathcal{D}_{p\to q}f(x) = \underbrace{\lim_{p\to q^+}\frac{f(x^p)-f(x^q)}{x^p-x^q}}_{\stackrel{\text{def}}{=}\mathbf{D}_{p\to q}[f(x)]} \tag{5.6}$$

$$\equiv \underbrace{\lim_{q\to p^-}\frac{f(x^p)-f(x^q)}{x^q-x^p}}_{\stackrel{\text{def}}{=}\mathbf{D}_{p\leftarrow q}[f(x)]} =: \lim_{q\to p^-}\mathcal{D}_{p\leftarrow q}f(x)$$

where $\mathbf{D}_{p\to q}[f(x)],\ \mathbf{D}_{p\leftarrow q}[f(x)]$ denote derivative through forward and backward (p,q)-power differences. Let us to show geometrical interpretation of (5.4) and (5.6)

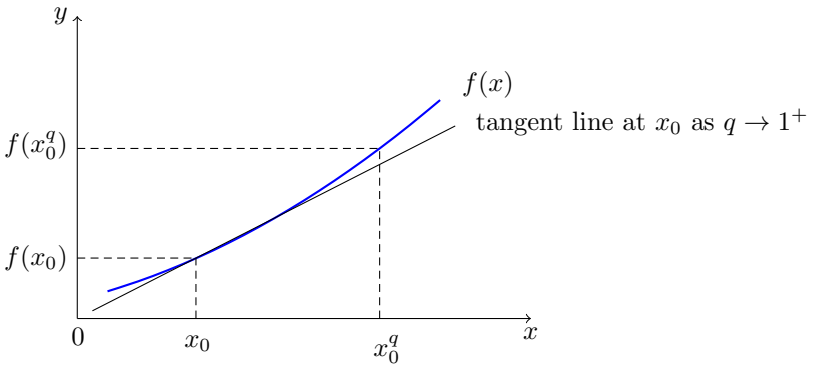

Figure 5. Geometrical sense of (5.4)

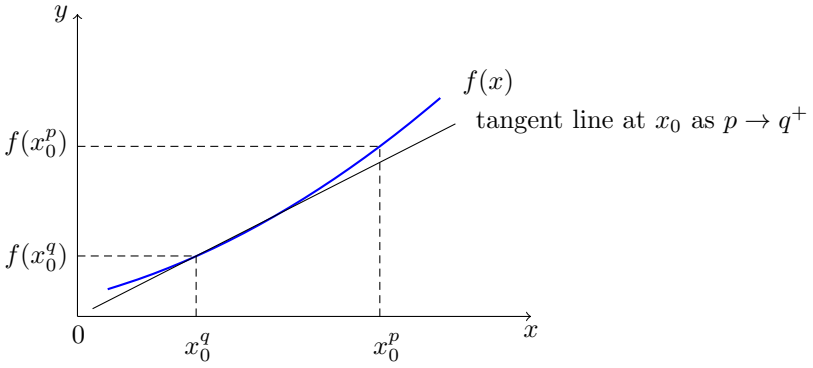

Figure 6. Geometrical sense of (5.6)

Applying (5.4) with monomial $x^m : m \in \mathbb{N}$, we get

$$\frac{d(x^m)}{dx} = \mathbf{D}_{q>1}[x^m] = \lim_{q\to 1^+}\left[\sum_{k=1}^{m}(x^q)^{m-k}\cdot x^{k-1}\right] = mx^{m-1} \tag{5.7}$$

$$\equiv \lim_{q\to 1^-}\mathbf{D}_{q<1}[x^m] = \lim_{q\to 1^-}\left[\sum_{k=1}^{m}x^{k-1}\cdot(x^q)^{m-k}\right] = mx^{m-1}$$

Note that $\mathbf{D}_{q<1}[x^m],\ \mathbf{D}_{q>1}[x^m]$ defined by (5.4). The high order $N \le m$ derivative, derived from (5.7)

$$\frac{d^N(x^m)}{dx^N} = \mathbf{D}^N_{q>1}[x^m] = \lim_{q\to 1^+}\prod_{j=0}^{N-1}\left(\sum_{k=1}^{m-j}(x^q)^{m-k}\cdot x^{k-j-1}\right) \tag{5.8}$$

$$\equiv \mathbf{D}^N_{q<1}[x^m] = \lim_{q\to 1^-}\prod_{j=0}^{N-1}\left(\sum_{k=1}^{m-j}x^{k-j-1}\cdot(x^q)^{m-k}\right)$$

Let be analytic function f and let f satisfies to Taylor's theorem 1.1 on segment of $(a,x),\ a\in\mathbb{R}$, then, applying (5.4), in radius of convergence of its Taylor's series, we obtain derivative

$$\frac{df(x)}{dx} = \sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}\mathbf{D}_{q<1}[(x-a)^k] \equiv \sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}\mathbf{D}_{q>1}[(x-a)^k] \tag{5.9}$$

Using $\mathbf{D}_{p\to q}[f(x)],\ \mathbf{D}_{p\leftarrow q}[f(x)]$ defined by (5.8), for each $v = x^p$, we receive

$$\frac{df(x)}{dx} = \left[\sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}\mathbf{D}_{p\to q}[(x-a)^k] \equiv \sum_{k=1}^{\infty}\frac{f^{(k)}(a)}{k!}\mathbf{D}_{p\leftarrow q}[(x-a)^k]\right]_{x=v} \tag{5.10}$$

Or, by means of definition (2.7) and (5.9), when $(x_0-a)\in\mathbb{N}$ derivative could be taken as follows

(5.11)
$$\frac{df(x)}{dx} = \sum_{k=1}^{\infty}\left\{\frac{f^{(k)}(a)}{k!}\cdot\lim_{n\to 1^+}\sum_{k=1}^{m}\xi(x-a,\ 1)_{nm-nk}x'\cdot\xi(x-a,\ 1)_{k-1}x'\right\}\Bigg|_{x=x_0}$$

Given x_0, such that $(x_0-a)\in\mathbb{R}^+$, then conditions of (3.2) is reached, and, applying definition (2.7), derivative f' follows

(5.12)
$$\frac{df(x)}{dx} = \sum_{k=1}^{\infty}\left\{\frac{f^{(k)}(a)}{k!}\cdot\lim_{n\to 1^+}\sum_{k=1}^{m}\xi(x-a,\ 1)_{nm-nk}x'\cdot\xi(x-a,\ 1)_{k-1}x'\right\}\Bigg|_{x=t_0}$$

Otherwise, let be $f : f\in C^n$, where n - positive integer, then under similar conditions as (5.11) and (5.13), derivative could be reached by differentiating of n-order Taylor's polynomial of f in terms of q-power difference (5.3) under limit notation over n

$$\frac{df(x)}{dx} = \sum_{k=1}^{n}\left\{\frac{f^{(k)}(a)}{k!}\cdot\lim_{n\to 1^+}\sum_{k=1}^{m}(x-a)^{nm-nk}x'\cdot(x-a)^{k-1}x'\right\} + R'_{n+1}(x) \tag{5.13}$$

Similarly, as (5.13), derivative of $f \in C^n$ in point $x = x_0$, such that $(x_0 - a) \in \mathbb{N}$

(5.14)
$$\frac{df(x)}{dx} = \sum_{k=1}^{n} \left\{ \frac{f^{(k)}(a)}{k!} \cdot \lim_{n \to 1^+} \sum_{k=1}^{m} \xi(x-a,\ 1)_{nm-nk} x' \cdot \xi(x-a,\ 1)_{k-1} x' \right\} \Bigg|_{x=x_0} + R'_{n+1}(x)$$

Otherwise, going from (5.14), $\forall (x_0 - a) \in \mathbb{R}^+$

(5.15)
$$\frac{df(x)}{dx} = \sum_{k=1}^{n} \left\{ \frac{f^{(k)}(a)}{k!} \cdot \lim_{n \to 1^+} \sum_{k=1}^{m} \xi(x-a,\ 1)_{nm-nk} x' \cdot \xi(x-a,\ 1)_{k-1} x' \right\} \Bigg|_{x=t_0} + R'_{n+1}(x)$$

6. Newton's interpolation formula

Being a discrete analog of Taylor's series, the Newton's interpolation formula [6], first published in his Principia Mathematica in 1687, hereby, by author's opinion, supposed to be discussed

$$f(x) = \sum_{k=0}^{\infty} \binom{x-a}{k} \Delta^k f(a) \tag{6.1}$$

Given $q = const$ in (2.3) divided q-difference $f[xq;\ x]$ is reached. Let be $\Delta f = f[xq;\ x](xq - x)$, then, by means of generalized high order forward finite difference $\Delta^k f,\ k \geq 2$, ([7], [8]), revised according to (2.3), Newton's formula (6.1) takes the form

$$f(x) = \sum_{k=0}^{\infty} \left[\binom{x-a}{k} \sum_{m=0}^{k} (-1)^m \binom{m}{k} f(x \cdot t^m) \right] \tag{6.2}$$

Review (5.4) and given $q = const$ divided q-power difference follows, by similar way as (6.2) reached, (6.1) could be written as

$$f(x) = \sum_{k=0}^{\infty} \left[\binom{x-a}{k} \sum_{m=0}^{k} (-1)^m \binom{m}{k} f(x^{n^{k-m}}) \right] \tag{6.3}$$

7. Conclusion

In this paper was discussed a way of obtaining real-valued smooth function's derivative in radius of convergence of it's Taylor's series or polynomial by means of analog of Newton's binomial theorem (1.20) in terms of q-difference (3.1) and (p, q)-power difference operators (5.12). In the last section reviewed a discrete analog of Taylor's series - Newton's interpolation formula (6.1), and applying operators of q-difference, (p, q)-power difference interpolation of initial function is shown (6.2), (6.3).

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