Workshop 2

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1 PROBLEM 1.3.10

The Cut Property of the real numbers is the following. If A and B are nonmempty, disjoint sets with A ∪ B = $\mathbb{R}$ and a < b for all $a \in A$ and $b \in B$, then there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.

(a) Use the Axiom of Completeness to prove the Cut Property
Proof: Suppose sets A and B are nonempty, disjoint sets with $A \cup B = \mathbb{R}$ and $a < b$ for all $a \in A$ and $b \in B$. We want to show there exists $c \in \mathbb{R}$ such that $x \leq c$ whenever $x \in A$ and $x \geq c$ whenever $x \in B$.
We know a<b $\forall a \in A$ and $\forall b \in B$. So, the set B is the set of upper bounds of set A. Therefore, A is bounded above and by the Axiom of Completeness, A contains a supremum (least upper bound) the we call s. Since $A \cup B = \mathbb{R}$ and $A \cap B = \emptyset$, either $s \in A$ and $s \notin B$ or $s \in B$ and $s \notin A$.
Case 1: $s \in A$ and $s \notin B$
$s \geq a \forall a \in A$ and $s < b \forall b \in B$
Case 2: $s \notin A$ and $s \in B$
$s \leq b \forall b \in B$ and $s > a \forall a \in A$
Therefore, $a \leq s \leq b \forall a \in A \forall b \in B$ and the Cut Property holds by the Axiom of Completeness.

(b) Show that the implication goes the other way; that is, assume $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. Prove sup E exists.
Assume that $\mathbb{R}$ possesses the Cut Property and let E be a nonempty set that is bounded above. We want to show that $\sup(E)$ exists. To do this, we have to show the two properties of a supremum.
(i) s is an upper bound for A

(ii) if b is any upper bound for A, then $s \leq b$.
Since E is bounded above and $\mathbb{R}$ has the Cut Property, then there exists a set F such that $E \cup F = \mathbb{R}$ and $E \cap F = \emptyset$. Because E is bounded above and $E \cup F = \mathbb{R}$, it implies that e < f $\forall e \in E \forall f \in F$. So F is the set of upper bounds for E. Also by the definition of the Cut Property, we have some $g \in \mathbb{R}$ such that $e \leq g \leq f \forall e \in E \forall f \in F$. Since $g \leq f$, $g \in F$ and it is the smallest element in F. Therefore, g is the least upper bound.

(c) The punchline of parts (a) and (b) is that the cut property could be used in place of the Axiom f Completeness as the fundamental axiom that distinguishes the real numbers from the rational numbers. To drive this point home, give a concrete showing that the Cut Property is not a valid statement when $\mathbb{R}$ is replaced by $\mathbb{Q}$.
The easiest example of this would be to let $A = a \in \mathbb{Q} : a^2 < 2$ and $B = b \in \mathbb{Q} : b^2 > 2$. From this, it is easy to see that $A \cap B = \emptyset$ and $A \cup B = \mathbb{Q}$. To find the "Cut Value" c, some simple arithmetic will show that $c^2 = 2$. We want to show that $a \leq c \leq b \forall a \in A \forall b \in B$. However, the value of c to solve this does not exist in $\mathbb{Q}$. Therefore, the Cut Property does not apply to the rational numbers.